WATER RESOURCES RESEARCH, VOL. 26, NO. 10, PAGES 2393–2400, OCTOBER 1990

Incorporating Hillslope Effects Into the Geomorphologic Instantaneous Unit Hydrograph

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Use of gamma distributions of stream holding times, rather than the traditional exponential distributions, results in geomorphologic instantaneous unit hydrographs (GIUHs) that better fit data-based IUHs. In this paper, hillslope effects are incorporated into the gamma GIUH (GGIUH) model by assuming that the hillslope travel distance in an area of a given order is approximated by the inverse of twice the local drainage density and introducing a hillslope velocity term. A method of moments fitting procedure is developed to estimate the channel and hillslope velocity terms in the GGIUH model from the moments of rainfall input and basin discharge output. It was found that hillslope velocities are 2 orders of magnitude smaller than channel velocities. The values found for the latter are reasonable given the range of values given in the literature for channel velocities. Similarly, the hillslope velocity term found by the method of moments procedure matches macropore velocities reported in the literature.

INTRODUCTION

Some recent developments in rainfall-runoff modeling show great promise in furthering the engineering and scientific goals of hydrologists. These developments include the incorporation of basin geomorphology into the river basin response function [*Boyd*, 1978; *Rodriguez-Iturbe and Valdes*, 1979; *Gupta et al.*, 1980; *Troutman and Karlinger*, 1985]. These models have the common feature of being relatively parsimonious in their data requirements. All the necessary geomorphological data can be obtained from topographic maps or from digital elevation model data. This paper presents a method of moments approach for considering and estimating separate velocity terms for the hillslope and for the channel components of the travel time in the geomorphological instantaneous unit hydrograph (GIUH) with gamma distributed stream lengths.

GEOMORPHOLOGIC INSTANTANEOUS UNIT HYDROGRAPH

In the last two decades, linear system theory has been the subject of considerable attention in rainfall-runoff modeling efforts [*Dooge*, 1973]. This theory holds that catchment response to a rainfall input is linear and time invariant [*Gupta et al.*, 1980]. Under this assumption, for a given effective rainfall intensity, $i(t)$, the key problem faced by the hydrologist is the determination of the instantaneous unit hydrograph (IUH). The output of the basin ($q(t)$: discharge at time t per unit area) is then given by the convolution integral of the IUH, $h(t)$, with the input:

$$q(t) = \int_0^t i(\tau)h(t-\tau)\, d\tau \qquad (1)$$

Paper number 90WR00862.
0043-1397/90/90WR-00862$05.00

Rodriguez-Iturbe and Valdes [1979] developed a probabilistic model relating the basin's characteristic response function, the IUH, to its geomorphology. Assuming a Strahler stream ordering system, the geomorphologic description of the basin is given by Horton's laws of stream numbers and streams lengths, and by Schumm's law of stream areas [*Horton*, 1945; *Strahler*, 1952; *Schumm*, 1956; *Smart*, 1972].

The GIUH developed by *Rodriguez-Iturbe and Valdes* [1979] was generalized for basins of any order by *Gupta et al.* [1980]. The latter development is summarized here. These authors showed that the GIUH of a basin is equal to the probability density function (pdf) of travel time to the catchment outlet, T_b, of a water drop falling randomly on the basin during a storm of uniform spatial distribution:

$$h(t) = \frac{d}{dt} P(T_b \le t) = \frac{d}{dt}\left[\sum_{s \in S} P(T_s \le t)P(s)\right] \qquad (2)$$

where S is the ensemble of all paths to the basin outlet, $P(s)$ is the probability of occurrence of path s, and T_s is the travel time in a path s. Expressed differently, the GIUH is the pdf of travel time along all possible paths to the basin outlet.

The probability $P(s)$ of following any given path to the outlet, in a basin of order Ω, is just the probability of starting out in the appropriate area of order i, times the probabilities of making each transition to streams of higher order along that path:

$$P(s) = \theta_i \times P_{ij} \times P_{jk} \times \cdots \times P_{1\Omega} \qquad (3)$$

where θ_i is referred to as the initial state probability. It is the probability of a random raindrop falling on areas contributing directly to streams of order i. P_{ij} is called the transition probability. It is the probability of a water drop going from a stream of order i to one of order j. The above probabilities can be shown to be strictly a function of the topology of the river basin. *Gupta et al.* [1980] give expressions that in an ensemble average sense yield the probabilities θ_i and P_{ij} in terms of the Horton's bifurcation and area ratios. However, in the appendix it is shown that in practice it is preferable to use actual data to derive θ_i and P_{ij} because the assumptions underlying the *Gupta et al.* [1980] equations are violated in river basins.

The travel time in a path s, T_s, is the sum of travel times in each element of that path:

$$T_s = T_{a(\omega)} + T_{r(\omega)} + \cdots + T_{r(\Omega)} \tag{4}$$

where $T_r(\omega)$ is the travel time in a stream of order $\omega(\omega\varepsilon[1, \cdots, \Omega])$. In the works by *Rodriguez-Iturbe and Valdes* [1979] and *Gupta et al.* [1980], it was assumed that the travel time from the hillslope to the stream is negligible ($T_{a(\omega)} = 0$). In the work presented in this paper, a framework is provided for incorporating nonnegligible travel times associated with hillslope effects into the GIUH.

Given that there are many streams of order ω, $T_{r(\omega)}$ can be considered an independent random variable with a given probability density function (pdf), $f_T^\omega(t)$. Thus from (2) the GIUH is given by the convolution of the individual pdfs, $f_T^\omega(t)$, for each order ω in a given path s, times the probability of that path, $P(s)$, summed over all paths to the outlet:

$$h(t) = \frac{d}{dt} P(T_b \le t) = \sum_{s \in S} [f_T^\omega(t) * \cdots * f_T^\Omega(t)] P(s) \tag{5}$$

where the asterisk is used to denote the convolution operation.

The distribution of travel times in streams of any given order ω, $f_T^\omega(t)$, has been frequently assumed to be exponential with parameter $\lambda_\omega = V/\bar{L}_\omega$, where $\bar{L}_\omega$ is the mean length of streams of order ω and V is some constant velocity characteristic of the basin [*Rodriguez-Iturbe and Valdes*, 1979; *Gupta et al.*, 1980].

Results from our analysis of stream and link lengths of simulated tree networks [*van der Tak*, 1988] and from three digitized channel networks suggest that a gamma pdf with shape parameter "a" of about 1.2 to 3.1 is more appropriate. These results corroborate those found by *Shreve* [1969], *Krumbein and Shreve* [1970] and *Smart* [1978], who suggest that a gamma shape parameter of 2 is suitable for the distribution of interior link lengths and a higher value is appropriate for exterior link lengths.

The gamma pdf is given by

$$f_{(a,b)}(L) = \frac{b^a L^{(a-1)} e^{-bL}}{\Gamma(a)} \tag{6}$$

where L is length, a is the shape parameter and b the scale parameter related through the mean by $\bar{L} = a/b$, and $\Gamma(a)$ is the gamma function. The variance of the gamma pdf is $\sigma_L^2 = a/b^2$.

It is easily shown that the pdfs of travel time have the same shape parameter as the pdfs of stream lengths by converting stream lengths to travel times by dividing through by a characteristic velocity V. However, the scale parameters of these gamma pdfs of travel time are given by $V \times b_L^\omega$, where b_L^ω is the scale parameter of the gamma pdf of stream lengths of order ω. Thus given a mean stream length for streams of order ω equal to $\bar{L}_\omega = a^\omega / b_L^\omega$ and an assumed gamma shape parameter a^ω, the scale parameter of $f_T^\omega(t)$ is

$$b_T^\omega = V a^\omega / \bar{L}_\omega \tag{7}$$

Given a complete Horton-Strahler analysis of the geomorphological characteristics of the basin, and hence the initial state and transition probabilities, θ_ω and P_{ij}, respectively, the only remaining parameters needed to determine the GIUH assuming a gamma distribution of stream travel times are the gamma shape parameters and the velocity. This version of the GIUH is hereinafter referred to as the GGIUH (gamma GIUH), whereas the version using exponentially distributed travel times is simply denoted GIUH.

van der Tak [1988] presents a detailed comparison of the GIUH and the GGIUH models, arguing for the preference of the latter. Figures 1*a* and 1*b* show the GIUH and GGIUH models for three different velocities, compared to IUHs derived by numerical deconvolution of rainfall-runoff data collected from two storm events in a channel network of Strahler order 5 in the 447 km^2 Souhegan River watershed in New Hampshire. The best fit of the peak Q_p and time to peak T_p is given in both cases by $V = 0.292$ m/s. However, the GGIUH (Figure 1*b*) provides a better match of the rising limb of the derived IUHs, particularly at very early times where the GIUH rises too quickly (Figure 1*a*). Even though the timing of the GIUH and GGIUH would corroborate the better fit obtained with the GGIUH model, Figures 1*a* and 1*b* show that selection of the velocity term is more important than the selection of the distribution of stream lengths.

Hillslope Effects and the Two Velocity Gamma GIUH

One might be tempted to assign a physical significance to the value of the velocity term V discussed previously by fitting the peak, Q_p, and the time to peak, T_p, of the GIUH and GGIUH models to those of the derived IUHs. This value of V incorporates both the hillslope and channel flow components of a water drop's path to the basin outlet because it reflects the basin's overall holding time as determined from the rainfall-runoff event data. Based on *Pilgrim*'s [1977] study, *Rodriguez-Iturbe and Valdes* [1979] postulated a constant characteristic velocity for the basin. *Nordin and Sabol* [1974] reviewed 51 time-of-travel tracer studies done by the U.S. Geological Survey and others on rivers with flows ranging from 17 to 34,000 cfs. In approximately half of these cases, convective velocities remained constant moving downstream; in a quarter the velocities tended to decrease in the downstream direction; and in the remainder, velocities either increased or no pattern was recognized. Given these observations it is important to point out that the GIUH and GGIUH models allow V to vary with stream order. A different V^ω can be used for each gamma pdf parameter b^ω in (7).

Rodriguez-Iturbe and Valdes [1979] ignored the hillslope component of overall travel time and argued that the holding time in the hillslope component of a water drop's path to the basin outlet is insignificant relative to the holding time in streams. They reasoned that the number of drops in a given stream segment which is derived directly from the adjacent hillslope is small compared to the number of drops from upstream tributaries to the stream. One might infer from this that the relative lengths of streams versus hillslope in a water drop's path is taken as evidence of the insignificance of the hillslope contribution to overall travel time. However, such

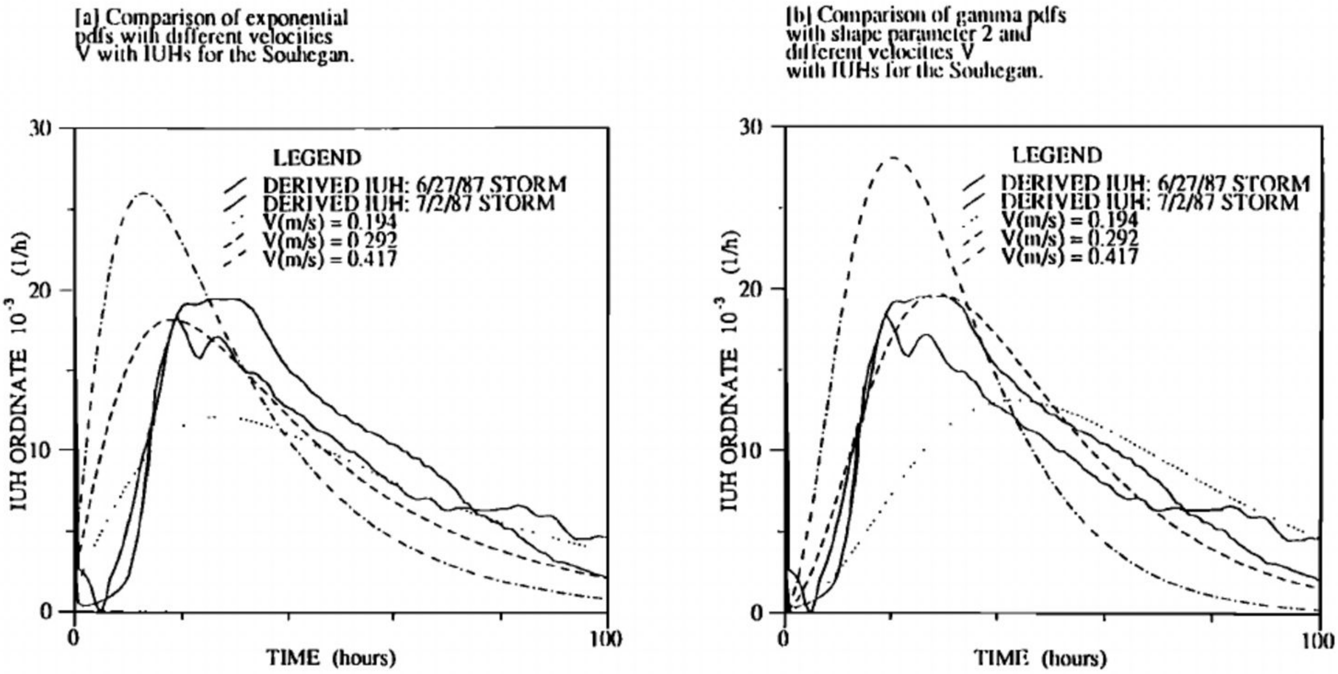

Fig. 1. Comparison of exponential and gamma GIUH fits to derived IUHs at different velocities: (*a*) comparison of exponential pdfs with different velocities V with IUHs for the Souhegan, and (*b*) comparison of gamma pdfs with shape parameter 2 and different velocities V with IUHs for the Souhegan.

an argument assumes that hillslope and channel velocities are of the same order of magnitude. This is a tenuous assumption.

We incorporate a hillslope velocity term V_h into the GGIUH in the following manner. A zeroth order stream of mean length equal to one half the inverse of the local drainage density [*Horton*, 1945] is included with a smaller velocity term ($V_h < V_c$) reflecting the different hillslope mechanisms governing flow to channels. These mechanisms, saturated overland flow, variable source areas, Hortonian overland flow, throughflow, and others, are beyond the scope of this work. Within the above framework the gamma scale parameter for the hillslope component in areas of a given order is given by

$$b_T^\omega = V_h a^\omega / \bar{L}_\omega = V_h a^\omega 2D_\omega \tag{8}$$

where D_ω is local drainage density given by

$$D_\omega = \frac{1}{N_\omega} \sum_{i=1}^{N_\omega} L_\omega^i / DA_\omega^i \tag{9}$$

where L_ω^i and DA_ω^i are the length and area contributing directly to stream i of order ω. Substituting V_c for V in (7) gives the gamma pdf scale parameter for the channel component. The two velocity GGIUH incorporating hillslope holding times is hereinafter designated 2VGGIUH.

Figures 2*a* and 2*b* show the effect of varying either the hillslope velocity, V_h, or the channel velocity, V_c, while the other term is held constant. The results are compared to a numerically derived IUH corresponding to the July 2, 1987, storm over the Souhegan River basin. Also shown is the best fit of the GGIUH with no hillslope component. The necessary aggregate "channel" velocity in that case is 0.292 m/s. Comparing Figures 2*a* and 2*b*, it appears that the 2VGGIUH model is more sensitive to the value of V_h than that of V_c. A doubling or quadrupling of the hillslope velocity V_h produces roughly twice the change in the peak of the 2VGGIUH, Q_p, than is seen for an equivalent relative increase in the channel velocity V_c.

The primary effect of incorporating the hillslope velocity into the GGIUH theory is to introduce a lag in the basin response. This is apparent in Figures 2*a* and 2*b* in which the one velocity GGIUH is seen to rise immediately from time zero while the 2VGGIUHs are not significantly different from zero until after about 5 hours. This lag is also apparent in the streamflow data-based IUH. As a result of this lag, the channel velocity term yielding a satisfactory fit to the derived IUH is considerably larger than that used in the one velocity GGIUH (compare the plots with $V_h = 0.0033$ m/s and $V_c = 0.75$ or 1.5 m/s in Figure 2*b* to the plot of the one velocity GGIUH with $V = 0.29$ m/s). The channel and hillslope velocities corresponding to the 2VGGIUH are reasonable given the range of values reported in the literature [*Nordin and Sabol*, 1974; *Beven and Germann*, 1982]. Rough measurements of V_c in the Souhegan gave values of about 0.6 m/s [*Wyss*, 1988].

In all of the 2VGGIUHs shown in Figure 2, distances traveled along the hillslope to the stream were assumed to be gamma distributed with shape parameter a of 2, just as we assumed for the distribution of stream lengths. However, there is no reason to believe that travel distances to streams exhibit the same probabilistic features as do stream lengths. *Wyss* [1988] found that horizontal distances to streams from grid points on a 25 × 25 m grid in the Souhegan were well represented by an exponential distribution. This assumption has a nonnegligible impact on the shape of the 2VGGIUH [*van der Tak*, 1988].

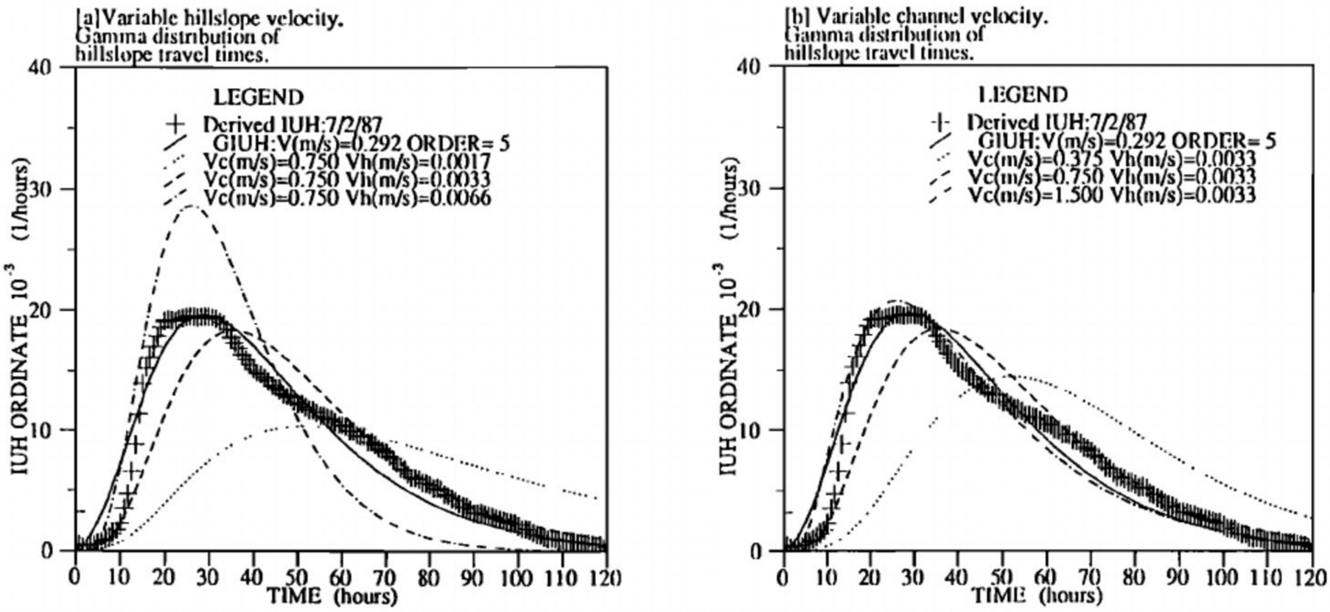

Fig. 2. The effect of varying hillslope or channel velocities in GGIUH models with reference to one velocity GGIUH and derived IUH. (*a*) Variable hillslope velocity. Gamma distribution of hillslope travel times. (*b*) Variable channel velocity. Gamma distribution of hillslope travel times.

Method of Moments Fits of the Two Velocity Gamma GIUH

Linear systems theory can be used to estimate moments of a river basin's response function. *Dooge* [1973] showed that the mean, μ_h, and variance, σ_h^2, of the IUH are given by

$$\mu_h = \mu_q - \mu_i \tag{10a}$$

$$\sigma_h^2 = \sigma_q^2 - \sigma_i^2 \tag{10b}$$

where μ_q and σ_q^2, and μ_i and σ_i^1 are the mean and variance of the discharge hydrograph and rainfall input, normalized to unit volume, respectively. Table 1 gives the moments of the normalized data from rainfall-runoff event on July 2, 1987, in the Souhegan and the estimated moments of the basin IUH.

The estimated moments of the basin response function given in Table 1 were used in conjunction with the moments of the 2VGGIUH of the Souhegan to derive V_h and V_c using a least squares approach. In order to accomplish this the functional relationship between the moments of the 2VGGIUH and the velocity parameters, V_h and V_c, had to be determined. The mean and variance of the 2VGGIUH were computed for a range of values of V_h and V_c for both the case where hillslope holding times are assumed to be exponential and that where they are assumed to be gamma with shape parameter 2. The relationships between the moments of the 2VGGIUH and V_c and V_h were found to be bivariate hyperbolic functions that are distinct depending on the assumptions for the distribution of hillslope holding times [*van der Tak*, 1988]. An equation of the form

$$\hat{Y} = AV_h^{-a} + BV_c^{-b} \tag{11}$$

was fit to the observed moments, where $\hat{Y}$ is the hyperbolic model estimate of the mean or the variance. The coefficients A, B, a and b were determined by minimizing the square of deviations. Because of the nonlinearity of this minimization problem, a good first estimate of the coefficients is required. Based on dimensional considerations, the exponents "a" and "b" were initially taken as 1 for the mean and 2 for the variance, respectively. Using these guesses, two sets of moment data were taken from each of the computed hyperbolic surfaces in order to estimate A and B. The resulting least squares hyperbolic fits for the case of exponential hillslope holding times are as follows:

$$\hat{\mu}_h = 0.211V_h^{-0.89} + 11.52V_c^{-1.04} \tag{12a}$$

$$\hat{\sigma}_h^2 = 0.126V_h^{-1.60} + 36.15V_c^{-2.17} \tag{12b}$$

The reasonableness of these equations is corroborated by Figures 3*a* and 3*b*. This shows the moments variation of the 2VGGIUH as a function of the hillslope velocity V_h for three different values of the channel velocity V_c, compared to the above hyperbolic equations for exponential hillslope holding times. The least squares hyperbolic equations fitted to the moments of the 2VGGIUH with gamma hillslope holding times are as follows:

$$\hat{\mu}_h = 0.1224V_h^{-0.99} + 12.38V_c^{-1.02} \tag{13a}$$

$$\hat{\sigma}_h^2 = 0.011V_h^{-1.92} + 46.73V_c^{-2.12} \tag{13b}$$

TABLE 1. Mean and Variance of Normalized Rainfall and Runoff From the July 2, 1987, Storm in the Souhegan, and Estimated Moments of the Basin Response Function From Equations (10*a*) and (10*b*)

	Mean, hours	Variance, hours^2
μ_q	52.93	
μ_i	5.21	
μ_h	47.72	
σ_q^2		538.95
σ_i^2		6.05
σ_h^2		532.90

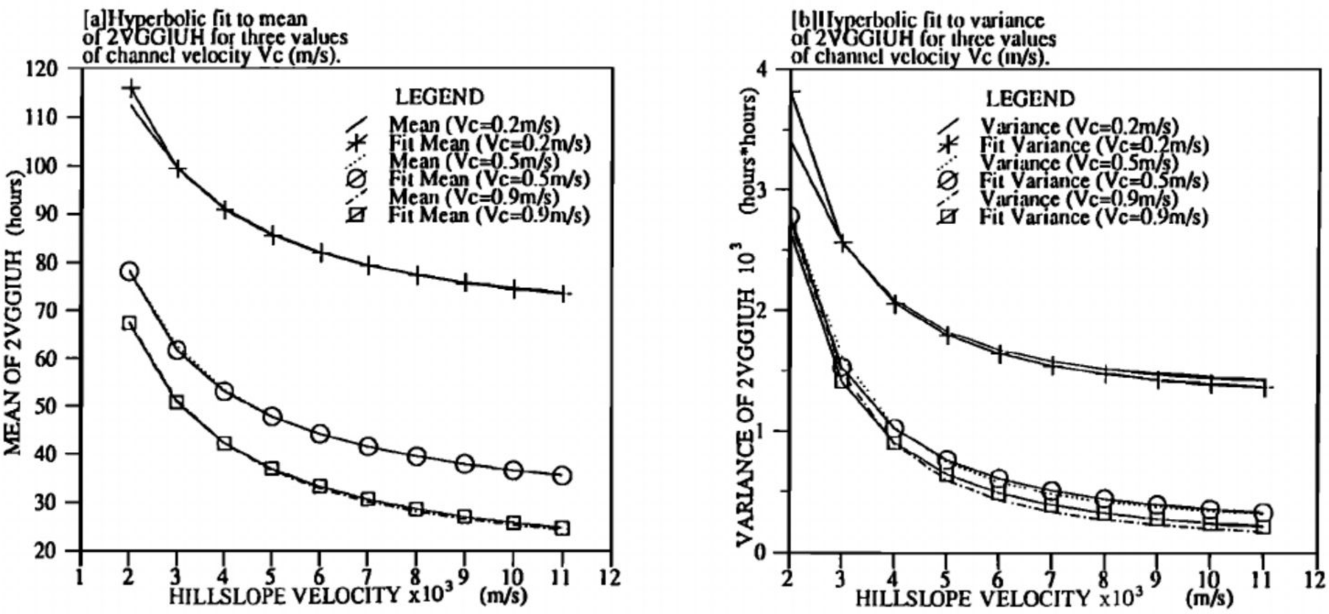

Fig. 3. Fit of bivariate hyperbolic equation to mean and variance of two velocity GGIUH with exponential pdf of hillslope holding times: (*a*) hyperbolic fit to mean of 2VGGIUH for three values of channel velocity *Vc*(m/s). (*b*) Hyperbolic fit to variance of 2VGGIUH for three values of channel velocity *Vc*(m/s).

The above gave values indistinguishable from the actual moments computed for the 2VGGIUH.

Using the above two sets of hyperbolic equations and the linear systems estimates for the moments of the basin response function ((10*a*) and (10*b*) and Table 1), we obtained least squares estimates of the hillslope and channel velocity terms.

The 2VGGIUH obtained by using the least squares estimates of V_h and V_c are shown in Figure 4*a* for both the case of exponential and gamma hillslope holding times. The difference in the values of V_h and V_c for the two cases is significant even though the 2VGGIUHs are not significantly different. Interestingly, the assumption of exponential hillslope pdfs yields a higher hillslope velocity than the gamma assumption while the channel velocity is only about 0.1 m/s larger than the best fit one velocity GGIUH value of V = 0.29 m/s. In contrast, the channel velocity, V_c, obtained for the gamma hillslope assumption is closer to the rough field estimates of $V_c \simeq 0.6$ m/s we have already mentioned [*Wyss*, 1988]. Obviously, the question of the appropriate parameterization of the distribution of hillslope holding times remains unresolved at this juncture.

In Figure 4*b* the actual moments of the data-based IUH of the July 2, 1987, storm were computed and used to fit V_h and

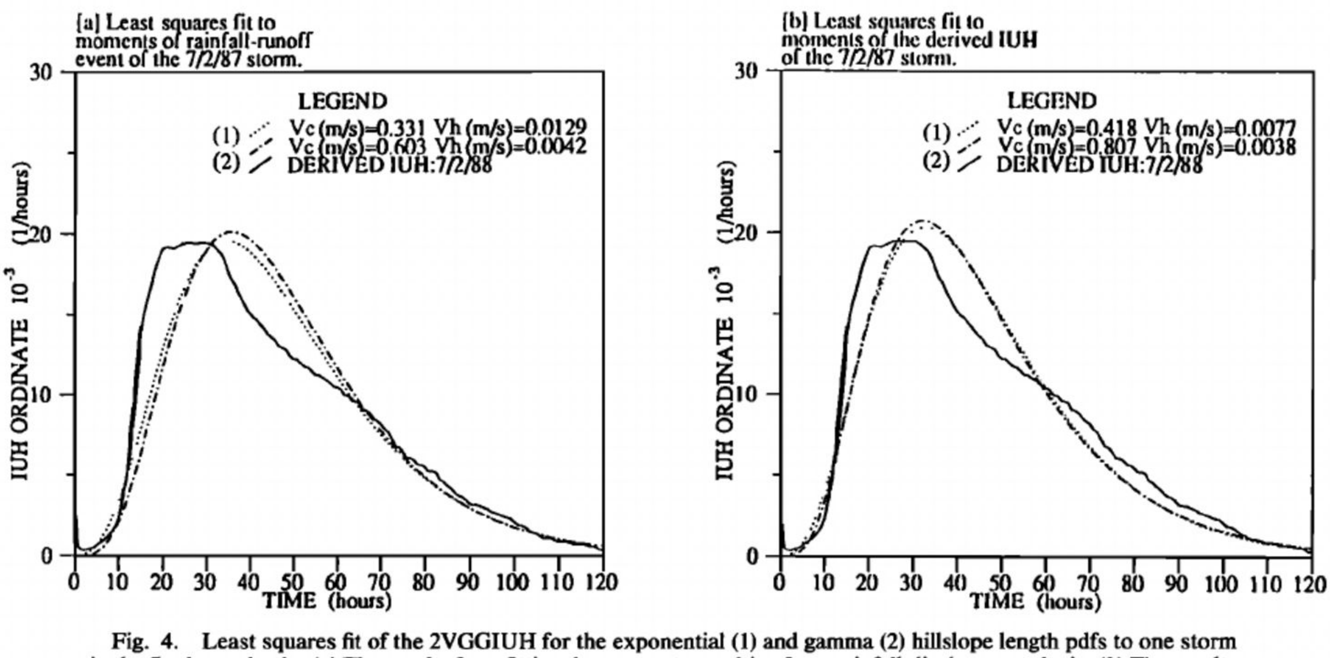

Fig. 4. Least squares fit of the 2VGGIUH for the exponential (1) and gamma (2) hillslope length pdfs to one storm in the Souhegan basin. (*a*) The results from fitting the moments resulting from rainfall-discharge analysis. (*b*) The results from directly fitting the moments of the IUH derived from the data via numerical deconvolution.

V_c, rather than using (10*a*) and (10*b*). Although improvement in the overall fit is seen, the estimate of the time to peak, T_p, is still about 10–20 hours behind the derived IUH, and the peak Q_p, is slightly overestimated.

Summary and Conclusion

Hillslope effects were incorporated into the GGIUH theory by assuming that the hillslope travel distance in an area of a given order is given on average by the inverse of twice the local drainage density and that velocities in the hillslope component of travel time are different than in channels. We presented a method of moments approach for estimating the hillslope and channel velocity parameters, V_h and V_c, from the moments of actual storm data. The resulting channel velocities are larger and more reasonable than those found by fitting the peak and time to peak of the derived IUH with the one velocity GIUH. The hillslope velocities were 2 orders of magnitude smaller than the channel velocities, which is in agreement with velocities reported in the literature for hillslope macropore flow mechanisms. Inclusion of the hillslope effects has a distinct effect on the shape of GGIUH by introducing a lag in the basin response function which more nearly approximates the behavior seen in the rising limb of derived IUHs and in actual hydrographs.

An unresolved issue resulting from our extensions of the GIUH theory includes the question of the appropriate parameterization of the holding times of the hillslope component of total travel time to the basin outlet. Are they exponentially distributed as suggested by *Wyss* [1988]? Or are they gamma distributed as is suggested by the relative values of the channel velocity terms resulting from the method of moments fits for the exponential and gamma pdf hillslope holding time assumptions used in this work?

Appendix: Determining Initial State and Transition Probabilities for the GIUH

Gupta et al. [1980] gave equations for estimating the initial state probabilities, θ_i, and the transition probabilities appearing in (3). The accuracy and assumptions of these formulas are the subject of this appendix.

Notes on the Initial State Probability Formula Given by Gupta et al. [1980]

Assuming a uniform rainfall over the watershed, the initial state probability θ_ω is determined exactly as the ratio of the total area draining directly into streams of order ω, DA_ω, to the total area of the basin A_Ω:

$$\theta_\omega = \frac{N_\omega \overline{DA}_\omega}{A_\Omega} \tag{A1}$$

where $\overline{DA}_\omega$ is the mean are draining directly into order ω streams, N_ω is the number of streams of order ω, and A_Ω is the total area of the basin.

Gupta et al. [1980] present the following useful recursive relationship to approximate θ_ω in a basin of order Ω:

$$\theta_1 = \frac{N_1 \bar{A}_1}{A_\Omega} \tag{A2a}$$

$$\theta_\omega = \frac{N_\omega}{A_\Omega}\left(\bar{A}_\omega - \sum_{j=1}^{\omega-1} \bar{A}_j (N_j P_{j\omega}/N_\omega)\right) \qquad \omega = 2, \cdots, \Omega \tag{A2b}$$

where $\bar{A}_\omega = (1/N_\omega)\sum_{i=1}^{N_\omega} A_{\omega_i}$ is the average area contributing to a stream of order ω and its tributaries, and A_{ω_i} is the area contributing to the *i*th stream of order ω and its tributaries.

There is no ambiguity in defining θ_1 by (A2*a*) above since the total area contributing directly to streams of order 1 (DA_1) is exactly $N_1\bar{A}_1$. However, (A2*b*) above, for the initial state probability of orders 2 through Ω, is slightly inaccurate. This inaccuracy is particularly disturbing in some cases where θ_ω is found to be negative.

Equation (A2*b*) is better understood when rewritten in the form

$$\theta_\omega = \frac{1}{A_\Omega}\left(N_\omega \bar{A}_\omega - \sum_{j=1}^{\omega-1} N_j \bar{A}_j P_{j\omega}\right) \qquad \omega = 2, \cdots, \Omega \tag{A3}$$

For this to be exact, the term in parentheses would have to be the exact area contributing directly to streams of order ω. Thus from the total cumulative area ($N_\omega \bar{A}_\omega$) we must subtract that proportion of the cumulative area contributing to order *j* streams ($j < \omega$) which drains into order ω streams. In other words, we subtract the area contributing to tributaries of order ω streams. To find the proportion of the area of order *j* streams which drains into order ω streams, hereinafter referred to as the tributary area, (A3) uses the transition probability $P_{j\omega}$. The transition probability $P_{j\omega}$ is defined as the number of order *j* streams draining into order ω streams divided by the total number of order *j* streams. The inaccuracy in (A3) arises from assuming that the area contributing to order *j* streams is uniformly distributed between different types of stream drainage paths. To illustrate the point, in an order 4 network, the area per order 2 streams which are tributaries to order 4 streams is probably smaller than the area per order 2 streams that flow into order 3 streams and then to order 4 streams. This is so since the 2 to 4 pathways are probably interior to the 2 → 3 → 4 pathways in the basin network. This is commonly observed in natural catchments.

Van der Tak [1988] illustrated the above point further by example, using simulated channel networks. In summary, the fundamental problem with applying (A2*a*) and (A2*b*) to a particular basin is that they incorrectly assume that the area drained by streams of order *j* is uniformly distributed between different drainage paths: $j \to i$, $i = j + 1, \cdots, \Omega$, where *i* and *j* refer to Strahler stream orders in the order Ω basin.

Notes on the Transition Probability Formula Given by Gupta et al. [1980]

The transition probability from Strahler order *j* streams to streams of order ω is, as was previously stated, the number of order *j* streams draining into order ω streams divided by the total number of order *j* streams. *Gupta et al.* [1980] calculated $P_{j\omega}$ with the following approximate relationship:

$$P_{j\omega} = \frac{(N_j - 2N_{j+1})E(\omega, \Omega)}{N_j \sum_{k=j+1}^{\Omega} E(k, \Omega)} + \frac{2N_{j+1}}{N_j}\,\delta_{j+1,\omega} \qquad \text{(A4)}$$

$$1 \le j < \omega \le \Omega$$

where $\delta_{j+1,\omega} = 1$ for $\omega = j + 1$, $\delta_{j+1,\omega} = 0$ otherwise; and $E(i,\Omega)$ is the mean number of links of order i in a basin of order Ω, given by [*Smart*, 1972]

$$E(i, \Omega) = N_i \prod_{j=2}^{i} (N_{j-1} - 1)/(2N_j - 1) \qquad i = 2, \cdots, \Omega \qquad \text{(A5)}$$

Equation (A4) contains assumptions which when applied to actual basins can lead to inaccuracies. The second term in expression (A4) is unambiguous. It accounts for those streams which combine to make a higher-order Strahler stream when $\omega = j + 1$. Recall that in the Strahler stream ordering scheme, when two streams of the same order come together the resulting channel downstream has a higher order. Thus in (A4) the proportion of streams of order j which do not combine to form higher-order streams is given by $(N_j - 2N_{j+1})/N_j$. $P_{j\omega}$ is then estimated as the product of this proportion of N_j streams with the estimated relative number of links of order ω in links downstream of order j streams, given by $E(\omega, \Omega)/\sum_{k=j+1}^{\Omega} E(k, \Omega)$. This assumption can lead to errors for two reasons. The primary reason is that it assumes, analogous to the assumptions for θ_ω given by (A2*b*), that stream transitions from streams of order j to streams of order i are uniformly distributed between different drainage pathways $j \rightarrow i$, $i = j + 1, \cdots, \Omega$, in direct proportion to the number of streams of each order j and i. A simple example serves to illustrate the inaccuracy of (A4).

Two hypothetical basins of order 3 with identical stream numbers but different drainage patterns are shown in Figure A1. Basin 1 has uniformly distributed stream drainages of order 1 (for those that do not create higher-order streams at their downstream confluence, hereinafter referred to as tributary-source, TS, links, following *Mock* [1971]), while basin 2 has all TS links draining into order 3 streams. The following table can be readily deduced for these basins.

Following the rationale for (A4), the following values of P_{13} are computed, using the actual link numbers given in Table A1.

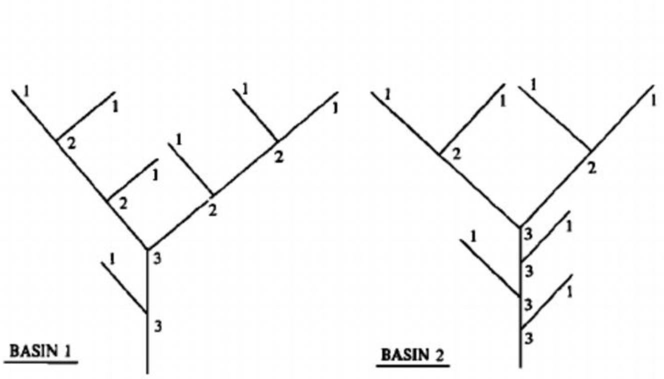

Fig. A1. Two basins with identical stream numbers but with different drainages of tributary-source links. Orders of links are indicated.

TABLE A1. Stream and Link Numbers for Basins 1 and 2 in Figure A1

Order	Stream Numbers	Link Numbers: Basin 1	Link Numbers: Basin 2
1	7	7	7
2	2	4	2
3	1	2	4

Basin 1 P_{13}(actual) $= \frac{1}{7}$ $\quad P_{13}$ (equation (A4)) $= \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$

Basin 2 P_{13}(actual) $= \frac{3}{7}$ $\quad P_{13}$ (equation (A4)) $= \frac{3}{7} \times \frac{4}{6} = \frac{2}{7}$

The rationale behind (A4) is seen to work for basin 1 where drainages of TS links are uniformly distributed between all streams of higher order, but it breaks down for basin 2 where the distribution is not uniform. This simple example shows that while (A4) does give a reasonable approximation to $P_{j\omega}$, it is a potential source of inaccuracy in the description of basin topology.

A further potential source of inaccuracy in (A4) results from the approximation inherent in the expression for $E(i, \Omega)$. This expression, discussed at length by *Smart* [1972], rests on the assumption of random topology in basin structure. The primary assumption is that of the independence of link lengths of different orders. This assumption seems questionable in practice. The expression for $E(i, \Omega)$ was found to be an inconsistent approximation of reality in several cases of actual and simulated basins, with deviations most often appearing for the highest order streams [*van der Tak*, 1988]. In the example given above, for the stream numbers in Table A1 we find $E(2, 3) = 2$ and $E(3, 3) = 0$.

In recent empirical studies by M. N. Allam and K. S. Balkhair (A geomorphologically-based runoff prediction model: Case study of two gauged watersheds in Saudi Arabia, unpublished manuscript, 1987) on geomorphologically based rainfall-runoff models, inaccuracies in θ_ω and $P_{j\omega}$ values of the sort described here were found when applying (A2) and (A4) to actual basins in Saudi Arabia. Given the relative ease of finding the exact values of these parameters with computer processing of digital cartographic data, it seems advisable for most practical applications to avoid using the approximations given by (A2) and (A4) in favor of measured values. This is particularly recommended in the case of channel networks exhibiting large deviations from random topology due, for instance, to geologic controls, in which case the assumptions of these equations are more severely violated. In the case of a relatively well-behaved channel network, such as the Souhegan River basin in southern New Hampshire, (A2) and (A4) can be adopted with insignificant effect on the shape of the GIUH.

Acknowledgment. This work was supported by National Science Foundation grant 8513556-ECE.

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(Received February 3, 1989;
revised March 26, 1990;
accepted April 11, 1990.)